Optimal regularized hypothesis testing in statistical inverse problems

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Computational Science and Engineering
Amsterdam, 3 March 2023

Structure

Introduction

Optimal regularized hypothesis testing

Adaptive hypothesis testing

Numerical results

Set-up

Consider statistical linear inverse problem

$$Y = Tu^\dagger + \sigma Z,$$

where

- $T\colon \mathcal{X} \to \mathcal{Y}$ bounded **linear forward operator** between real Banach space $\mathcal{X}$ and real Hilbert space $\mathcal{Y}$,
- $u^\dagger \in \mathcal{X}$ unknown quantity of interest,
- $\sigma > 0$ noise level,
- Z **Gaussian white noise** process on $\mathcal{Y}$.

Estimation and inference of features

- $\mathcal{X}$,$\mathcal{Y}$ typically function spaces such as $L^p(\Omega)$ or $H^s(\Omega)$ on some domain $\Omega \subseteq \mathbb{R}^d$.
- Often one is not interested in whole function $u^\dagger$ but in **certain features** of it such as modes, homogeneity, monotonicity, or support.
- Many features can be described by (family of) bounded linear functionals $\varphi \in \mathcal{X}^*$.
- We perform inference for such features by means of statistical hypothesis testing. Specifically, we test

$$H_0 : \left\langle \varphi, u^\dagger \right\rangle_{\mathcal{X}^* \times \mathcal{X}} = 0 \quad \text{against} \quad H_1 : \left\langle \varphi, u^\dagger \right\rangle_{\mathcal{X}^* \times \mathcal{X}} > 0.$$

Unregularized hypothesis testing

- For $\varphi \in \operatorname{ran} T^*$, construct **unregularized test**

$$\Psi_0(Y) := \mathbf{1}_{\langle Y, \Phi_0 \rangle > c_0}$$

using probe element $\Phi_0 \in \mathcal{Y}$ such that

$$\varphi = T^* \Phi_0.$$

- Critical value c_0 can be chosen such that Ψ_0 has prescribed **level of significance** $\alpha \in (0, 1)$.
- Ψ_0 has certain optimality properties [Proksch, Werner, Munk 2018].

Problems and solutions

Drawbacks

1. Unregularized test Ψ_0 **not defined** for $\varphi \notin \operatorname{ran} T^*$.
2. Probe element Φ_0 is solution to **ill-posed equation**

$$T^*\Phi_0 = \varphi.$$

As a consequence, **power** of Ψ_0 can be **arbitrarily close to level** α.

Our contribution

We resolve both of these issues by **maximizing** the **power** among a class of level-α tests based upon linear estimators.

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A class of level-α tests

- Consider test
$$\Psi_\Phi(Y) := \mathbf{1}_{\langle Y,\Phi\rangle > c}$$
with **arbitrary** probe element $\Phi \in \mathcal{Y}$.

Assumptions

There exists a pair $(\mathcal{V}, \mathcal{V}')$ of Banach spaces such that

1. $\langle v', v\rangle_{\mathcal{X}^*\times\mathcal{X}} \leq \|v'\|_{\mathcal{V}'} \|v\|_{\mathcal{V}}$ for all $v \in \mathcal{V} \cap \mathcal{X}$, $v' \in \mathcal{V}' \cap \mathcal{X}^*$,
2. $u^\dagger \in \mathcal{V} \cap \mathcal{X}$ with $\|u^\dagger\|_{\mathcal{V}} \leq 1$,
3. $\operatorname{ran} T^* \subseteq \mathcal{V}'$ and $T^*\colon \mathcal{Y} \to \mathcal{V}'$ is bounded,
4. $\varphi \in \overline{\operatorname{ran} T^*}$.

- For any $\Phi \in \mathcal{Y}$, critical value c can be chosen such that Ψ_Φ has **at most level** $\alpha \in (0, 1)$ under these assumptions.

Optimal regularized hypothesis testing

- For any $\Phi \in \mathcal{Y} \setminus \{0\}$, Ψ_Φ has **power**

$$\mathbb{P}_{u^\dagger}\left[\Psi_\Phi(Y) = 1\right] = Q\left(q_\alpha^{\mathcal{N}} - \frac{\|T^*\Phi - \varphi\|_{\mathcal{V}'} - \langle Tu^\dagger, \Phi\rangle_{\mathcal{Y}}}{\sigma\,\|\Phi\|_{\mathcal{Y}}}\right),$$

  where Q and $q_\alpha^{\mathcal{N}}$ are the cdf and α-quantile of $\mathcal{N}(0,1)$.
- Find optimal probe element $\Phi^\dagger$ as minimizer of $J_{Tu^\dagger}^{\mathcal{Y}}$,

$$J_{Tu^\dagger}^{\mathcal{Y}}(\Phi) := \frac{\|T^*\Phi - \varphi\|_{\mathcal{V}'} - \langle Tu^\dagger, \Phi\rangle_{\mathcal{Y}}}{\|\Phi\|_{\mathcal{Y}}}.$$

  Then $\Psi_{\Phi^\dagger}$ has **maximal power** among $\{\Psi_\Phi : \Phi \in \mathcal{Y}\}$.

Problem

In practice $u^\dagger$ is unknown, so $J_{Tu^\dagger}^{\mathcal{Y}}$ is unaccessible.

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Adaptive hypothesis testing

- Choose probe element Φ as minimizer of $J_Y^{\mathcal{Z}}$,

$$J_Y^{\mathcal{Z}}(\Phi) := \frac{\|T^*\Phi - \varphi\|_{\mathcal{Y}'} - \langle Y, \Phi\rangle_{\mathcal{Z}^*\times\mathcal{Z}}}{\|\Phi\|_{\mathcal{Z}}},$$

  where $\mathcal{Z} \subset \mathcal{Y}$ is dense continuously embedded subspace such that data Y is **bounded** linear functional on $\mathcal{Z}$.
- Use two independent samples Y_1 and Y_2 of data, one to construct test, another to evaluate test on.
- Define **adaptive test**

$$\Psi^*(Y_2; Y_1) := \begin{cases} \Psi_\Phi(Y_2) & \text{if } J_{Y_1}^{\mathcal{Z}} \text{ has global minimizer } \Phi \in \mathcal{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

  Then Ψ^* has **level** α.

Computability

- Define **convex** surrogate functional $\hat{J}_Y^{\mathcal{Z}} \colon \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$,

$$\hat{J}_Y^{\mathcal{Z}}(e, s) := \| T^* e - s\varphi \|_{\mathcal{V}'} - \langle Y, e \rangle_{\mathcal{Z}^* \times \mathcal{Z}} .$$

- If $(e, s) \in \mathcal{Z} \times \mathbb{R}$ is solution to

$$\min \hat{J}_Y^{\mathcal{Z}}(e, s) \qquad \text{subject to} \qquad \|e\|_{\mathcal{Z}} \leq 1, \quad s \geq 0$$

with $e \neq 0$, $s > 0$, then

$$\Phi = s^{-1} e$$

is **global minimizer** of $J_Y^{\mathcal{Z}}$.

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Numerical simulations: Deconvolution

- Consider convolution operator $Tu = h * u$ on $\mathcal{X} = \mathcal{Y} = L^2(\mathbb{R})$ with kernel $h \in L^1(\mathbb{R})$ given by

$$(\mathcal{F}h)(\xi) = \left(1 + 0.0009\xi^2\right)^{-2} \quad \text{for all } \xi \in \mathbb{R}.$$

- **Question:** Is $\operatorname{supp} u^\dagger \cap [0, l] = \emptyset$?
- Choose

$$\mathcal{V} := L^1(\mathbb{R}), \quad \mathcal{V}' := L^\infty(\mathbb{R}), \quad \mathcal{Z} = H^{0.51}(\mathbb{R}).$$

- Choose critical value of all tests such that **level** is

$$\alpha = 0.1.$$

Considered scenarios

Choose φ and $u^\dagger$ as β-**kernels**.

(S1) Compatible smooth scenario

(S2) Compatible nonsmooth scenario

(S3) Incompatible smooth scenario

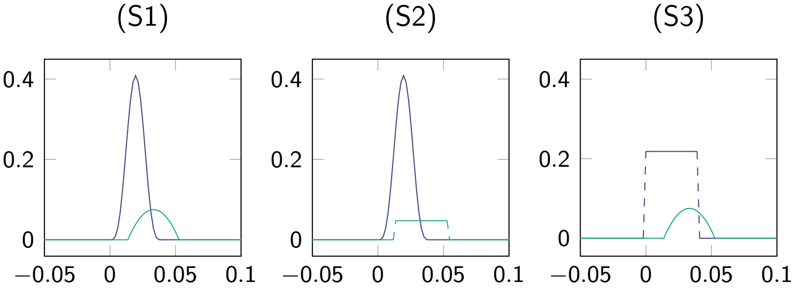

Figure: The function φ (——) and the truth $u^\dagger$ (——).

Results – compatible smooth scenario (S1)

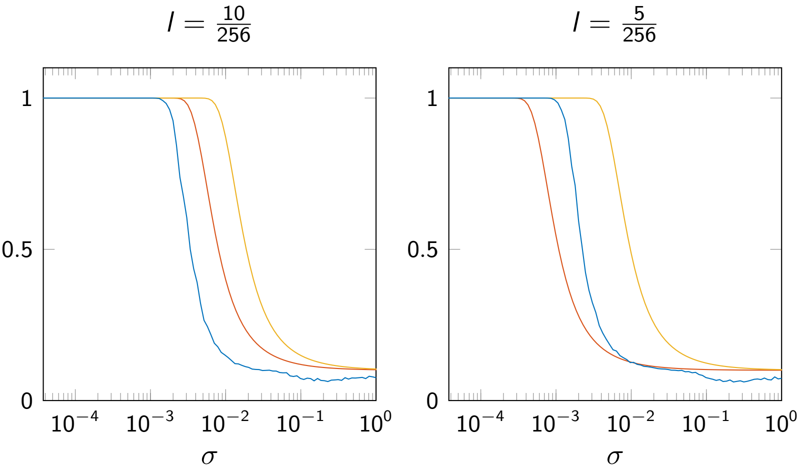

Figure: Exact power of unregularized test (——), optimal test (——), and empirical power of adaptive test (——) based upon 100 samples.

Results – compatible nonsmooth scenario (S2)

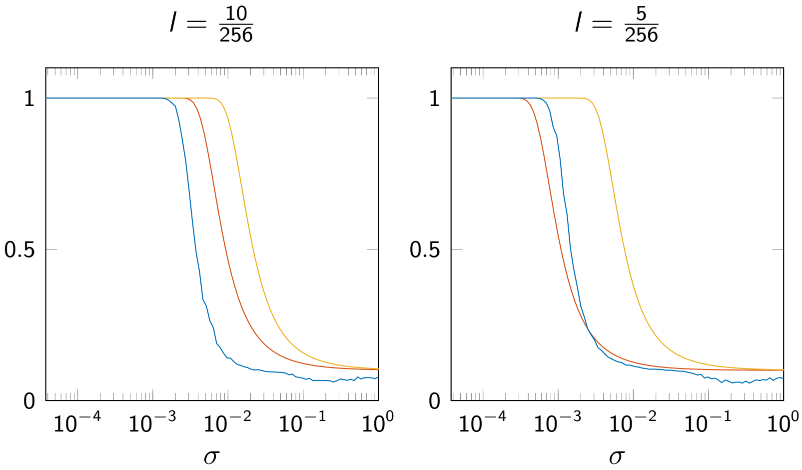

Figure: Exact power of unregularized test (——), optimal test (——), and empirical power of adaptive test (——) based upon 100 samples.

Results – incompatible smooth scenario (S3)

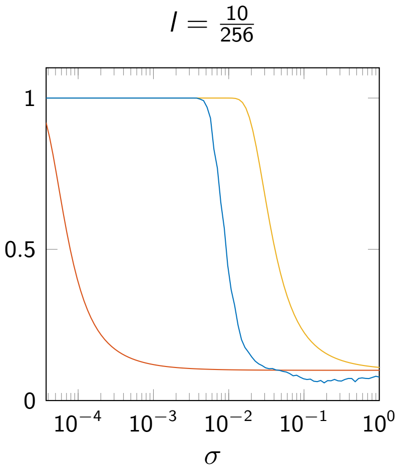

Figure: Exact power of unregularized test (——), optimal test (——), and empirical power of adaptive test (——) based upon 100 samples.

Conclusion

- For given feature $\varphi \in \overline{\text{ran } T^*}$, optimal level-$\alpha$ test based upon linear estimator exists under a priori assumptions on $u^\dagger$.
- Adaptive test can be constructed by solving constrained convex optimization problem.
- Adaptive test allows testing of features for which unregularized testing is unfeasable due to ill-posedness.

Outlook

- Study power of adaptive test for other problems.
- Tikhonov-regularized hypothesis testing

References

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