Non-asymptotic error estimates for the Laplace approximation in Bayesian inverse problems

Tapio Helin **Remo Kretschmann**

LUT University

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Structure

Focus

Approximation of posterior distribution of Bayesian inverse problem by **Gaussian distribution** according to Laplace's method.

Laplace's method

1. Replace log-posterior density by second order Taylor approximation around MAP estimate
2. Renormalise

Examples for use of Laplace approximation

- When **sampling** posterior distribution is **too expensive**.
- Inverse problems that are **close to linear problem**.

Motivation

- **Asymptotic properties** of Laplace approximation in small noise or large data limit have been studied extensively.
- In practice, one is often interested in **quantifying approximation error** for **given noise level**.

Problems

Nonlinearity of problem or **high problem dimension** may cause **large approximation error** even for low noise level.

Goal

Understand and **quantify influence of**

1. **nonlinearity of forward mapping**,
2. **problem dimension**

on Laplace approximation error.

Contribution

Main results

Non-asymptotic error estimates in total variation distance for Laplace approximation in Bayesian inverse problems:

1. Central error estimate
2. Error estimate that **makes explicit** influence of **non-Gaussianity of likelihood**, **non-Gaussianity of prior**, and **problem dimension**
3. Error estimate for **perturbed linear problems** with Gaussian prior that **makes explicit** influence of **nonlinear perturbation**

Total variation error of Laplace approximation is **measure of non-Gaussianity** of posterior distribution.

Structure

Set-up

For $\varepsilon > 0$, recover $x \in \mathbb{R}^d$ from noisy measurement $y \in \mathbb{R}^d$, where

$$y = G(x) + \sqrt{\varepsilon}\eta.$$

- Nonlinear **forward mapping** G
- **Random noise** $\eta \in \mathbb{R}^d$ with standard normal distribution

$$\eta \sim \mathcal{N}(0, I_d)$$

- **Prior distribution**

$$\mu(\mathrm{d}x) = \exp(-R(x))\mathrm{d}x$$

- **Posterior distribution** according to Bayes' formula given by

$$\mu^y(\mathrm{d}x) \propto \exp\left(-\frac{1}{2\varepsilon}|y - G(x)|^2 - R(x)\right)\mathrm{d}x$$

Laplace approximation

Assumption

$$I(x) := \frac{1}{2}|y - G(x)|^2 + \varepsilon R(x)$$

has **unique minimiser** $\hat{x} \in \mathbb{R}^d$, $I \in C^2(\mathbb{R}^d, \mathbb{R})$, and Hessian $(\mathrm{H}I)(\hat{x})$ is **positive definite**.

Then, **Laplace approximation** of μ^y defined as

$$\mathcal{L}_{\mu^y} := \mathcal{N}(\hat{x}, \varepsilon\Sigma),$$

where $\Sigma := (\mathrm{H}I(\hat{x}))^{-1}$. This way,

$$\mu^y(\mathrm{d}x) \propto \exp\left(-\frac{1}{\varepsilon}I(x)\right)\mathrm{d}x,$$

$$\mathcal{L}_{\mu^y}(\mathrm{d}x) \propto \exp\left(-\frac{1}{2\varepsilon}\|x - \hat{x}\|_\Sigma^2\right)\mathrm{d}x.$$

Assumptions

Define $\Phi(x) := \frac{1}{2}|y - G(x)|^2$, so that $I(x) = \Phi(x) + \varepsilon R(x)$.

Bounds on log-likelihood and log-prior density

$\Phi, R \in C^3(\mathbb{R}^d, \mathbb{R})$ and there exists $K > 0$ such that

$$\max\left\{\|D^3\Phi(x)\|_\Sigma, \|D^3R(x)\|_\Sigma\right\} \leq K \quad \text{for all } x \in \mathbb{R}^d,$$

where $\|D^3\Phi(x)\|_\Sigma := \sup\left\{|D^3\Phi(x)(h_1, h_2, h_3)| : \|h_j\|_\Sigma \leq 1\right\}$.

Quadratic bound on log-posterior density

There exists $0 < \delta \leq 1$ such that

$$I(x) - I(\hat{x}) \geq \frac{\delta}{2}\|x - \hat{x}\|_\Sigma^2 \quad \text{for all } x \in \mathbb{R}^d.$$

Central error estimate

Theorem

Under previous assumptions on Φ, R, and I, we have

$$d_{\mathrm{TV}}(\mu^y, \mathcal{L}_{\mu^y}) \leq E_1(r_0; K) + E_2(r_0; \delta) \quad \text{for all } r_0 \geq 0,$$

where

$$E_1(r_0; K) := (2\varepsilon)^{-\frac{d}{2}} \frac{2}{\Gamma\left(\frac{d}{2}\right)} \int_0^{r_0} f(r) r^{d-1} \mathrm{d}x,$$

$$E_2(r_0; \delta) := \delta^{-\frac{d}{2}} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \Gamma\left(\frac{d}{2}, \frac{\delta r_0^2}{2\varepsilon}\right),$$

and

$$f(r) := \left[\exp\left(\frac{(1+\varepsilon)K}{6\varepsilon} r^3\right) - 1\right] \exp\left(-\frac{1}{2\varepsilon} r^2\right).$$

Optimal choice of r_0

Proposition

Optimal choice of r_0 in previous estimate is either 0 or satisfies

$$\exp\left(\frac{(1+\varepsilon)K}{6\varepsilon} r_0^3\right) - 1 - \exp\left(\frac{1-\delta}{2\varepsilon} r_0^2\right) = 0.$$

Structure

Explicit error estimate

Theorem

Suppose that previous assumptions on Φ, R, and I hold. If K, δ, ε, and d satisfy

$$\frac{6\delta^{\frac{3}{2}}}{(1+\varepsilon)\varepsilon^{\frac{1}{2}}K} \geq \max\left\{8d^{\frac{3}{2}}, \left(8\ln\frac{2}{C(1+\varepsilon)\varepsilon^{\frac{1}{2}}K\delta^{\frac{d}{2}}C_d}\right)^{\frac{3}{2}}\right\}$$

where

$$C := \frac{1}{3}\sqrt{2e} \quad \text{and} \quad C_d := \frac{\Gamma\left(\frac{d}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},$$

then

$$d_{\mathrm{TV}}(\mu^y, \mathcal{L}_{\mu^y}) \leq 2C(1+\varepsilon)\varepsilon^{\frac{1}{2}}KC_d.$$

- Note that $C_d \asymp \left(\frac{d}{2}\right)^{\frac{3}{2}}$ as $d \to \infty$.

Asymptotic behaviour as problem dimension $d \to \infty$

Index K_d, δ_d, and ε_d by $d \in \mathbb{N}$.

Corollary

Suppose that previous assumptions hold for all $d \in \mathbb{N}$. If
$\delta_d \leq e^{-1/2}$, $\varepsilon_d \leq 1$,

$$\varepsilon_d^{\frac{1}{2}} K_d \to 0, \quad \text{and} \quad \frac{3}{\varepsilon_d^{\frac{1}{2}} K_d} \geq \left(\frac{8}{\delta_d} \ln \frac{1}{\delta_d} \right)^{\frac{3}{2}} d^{\frac{3}{2}}$$

for all $d \in \mathbb{N}$, then for every $C > \frac{1}{3} e$ there exists $N \in \mathbb{N}$ such that

$$d_{\mathrm{TV}}(\mu^y, \mathcal{L}_{\mu^y}) \leq C \varepsilon_d^{\frac{1}{2}} K_d d^{\frac{3}{2}}$$

for all $d \geq N$.

Structure

Introduction

Central error estimate

Explicit error estimate

Perturbed linear problems with Gaussian prior

Perturbed linear problems with Gaussian prior

- **Forward mapping** G given by linear mapping with small nonlinear perturbation

$$G_\tau(x) = Ax + \tau F(x),$$

where $A \in \mathbb{R}^{d\times d}$, $F \in C^3(\mathbb{R}^d)$, and $\tau \geq 0$.
- **Gaussian prior distribution** $\mu = \mathcal{N}(m_0, \Sigma_0)$

Assumption

There exists $\tau_0 > 0$, such that for all $\tau \in [0, \tau_0]$,

$$I_\tau(x) = \frac{1}{2}|Ax + \tau F(x) - y_\tau|^2 + \frac{\varepsilon}{2}\|x - m_0\|^2_{\Sigma_0}$$

has **unique minimiser** $\hat{x}_\tau$ with $(\mathrm{H}I_\tau)(\hat{x}_\tau) > 0$. Furthermore, y_τ, $\hat{x}_\tau$, and $\Sigma_\tau := \mathrm{H}I_\tau(\hat{x}_\tau)^{-1}$ **converge** as $\tau \to 0$ with $\lim_{\tau\to 0} \Sigma_\tau > 0$.

Assumptions

Let $B(M) \subset \mathbb{R}^d$ denote a closed Euclidean ball with radius M around the origin.

Bounds on nonlinear perturbation

There exist $C_0, \ldots, C_3 > 0$ and $M > 0$ such that

$$\|D^j F(x)\|_{\Sigma_\tau} \leq C_j, \quad j = 0, \ldots, 3,$$

for all $x \in \mathbb{R}^d$ and $\tau \in [0, \tau_0]$, and

$$D^3 F \equiv 0 \quad \text{on } \mathbb{R}^d \setminus B(M).$$

Error estimate for perturbed linear problems

Theorem

Under the previous assumptions, there exists $\tau_1 \in (0, \tau_0]$ such that

$$d_{\mathrm{TV}}(\mu^{y_\tau}, \mathcal{L}_{\mu^{y_\tau}}) \leq 2CC_d(1+\varepsilon)\varepsilon^{\frac{1}{2}} \left(V(\tau)\tau + \frac{W}{2}\tau^2 \right)$$

for all $\tau \in [0, \tau_1]$, where

$$C := \frac{1}{3}\sqrt{2}e, \quad C_d := \frac{\Gamma\left(\frac{d}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},$$

$$V(\tau) := C_3(\|A\|M + |y_\tau|) + 3C_2 \left\| A\Sigma_\tau^{\frac{1}{2}} \right\|,$$

$$W := C_3C_0 + 3C_2C_1.$$

Moreover, $\{V(\tau)\}_{\tau \in [0,\tau_1]}$ is bounded.

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